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(36) 11.1 11.2 Calculus with parametric curves

Example 1 31 August 2008 A

A curve C is given by the parametric equations

$x = \sin t + \cos t$ and $y = \sin t - \cos t$, where $-\pi \le t \le \pi$.

(a) Find the points $P(x, y)$ on C where it has horizontal or vertical tangent lines.
(b) Find the length of the curve C

Solution

$$x = \sin t + \cos t \qquad \frac{dx}{dt} = \cos t - \sin t$$

$$y = \sin t - \cos t \qquad \frac{dy}{dt} = \cos t + \sin t$$

$$m = \frac{dy}{dt} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\cos t + \sin t}{\cos t - \sin t}$$

$H.T$

$$\frac{dy}{dt} = 0 \quad , \frac{dx}{dt} \neq 0$$

$$\cos t + \sin t = 0$$
$$\sin t = -\cos t$$
$$\tan t = -1$$
$$\alpha = \frac{\pi}{4}$$

$\boxed{2}$

$$t = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$\boxed{4}$

$$t = -\alpha = -\pi$$

at $t = \frac{3\pi}{4}$

$$x|_{t=\frac{3\pi}{4}} = \sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} = 0$$

$$y|_{t=\frac{3\pi}{4}} = \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

at $t = -\frac{\pi}{4}$

$$x|_{t=-\frac{\pi}{4}} = \sin\left(\frac{-\pi}{4}\right) + \cos\left(\frac{-\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$$

$$y|_{t=-\frac{\pi}{4}} = \sin\left(\frac{-\pi}{4}\right) - \cos\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

$$\therefore \text{H.T at} \quad (0, \sqrt{2}) \;, \quad (0, -\sqrt{2})$$

$$V.T \quad \frac{dx}{dt} = 0 \quad , \quad \frac{dy}{dt} \neq 0$$

$$\cos t - \sin t = 0$$
$$\sin t = \cos t$$
$$\tan t = 1$$
$$\alpha = \frac{\pi}{4}$$

$\boxed{1}$

$$t = \alpha = \frac{\pi}{4}$$

$\boxed{3}$

$$t = -\pi + \alpha = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

$at \quad t = \frac{\pi}{4}$

$$x|_{t=\frac{\pi}{4}} = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$y|_{t=\frac{\pi}{4}} = \sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$at \quad t = \frac{-3\pi}{4}$

$$x|_{t=\frac{-3\pi}{4}} = \sin\left(\frac{-3\pi}{4}\right) + \cos\left(\frac{-3\pi}{4}\right) = \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} = -\sqrt{2}$$

$$y|_{t=\frac{-3\pi}{4}} = \sin\left(\frac{-3\pi}{4}\right) - \cos\left(\frac{-3\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{-1}{\sqrt{2}} = 0$$

$$\therefore \text{V.T at } (\sqrt{2}, 0) \quad , \quad (-\sqrt{2}, 0)$$

(b)

$$\left(\frac{dx}{dt}\right)^2 = (\cos t - \sin t)^2 = \cos^2 t + \sin^2 t - 2\sin t \cos t = 1 - 2\sin t \cos t$$

$$\left(\frac{dy}{dt}\right)^2 = (\cos t + \sin t)^2 = \cos^2 t + \sin^2 t + 2\sin t \cos t = 1 + 2\sin t \cos t$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = 2$$

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \; dt$$

$$L = \int_{-\pi}^{\pi} \sqrt{2} \; dt = \sqrt{2} \left[t \right]_{-\pi}^{\pi} = \sqrt{2}\,[\pi - (-\pi)] = \sqrt{2}\,[\pi + \pi] = 2\sqrt{2}\,\pi$$

Example 2

33 June 2009 A

Find the surface area generated by rotating the parametric curve
$x = \cos t + \sin t$, $y = \sin t - \cos t$, $0 \le t \le \pi/2$ around the $y -$ axis.

Solution

$$x = \cos t + \sin t \qquad \frac{dx}{dt} = -\sin t + \cos t$$

$$y = \sin t - \cos t \qquad \frac{dy}{dt} = \cos t + \sin t$$

$$\left(\frac{dx}{dt}\right)^2 = \cos^2 t + \sin^2 t - 2\sin t\cos t = 1 - \sin 2t$$

$$\left(\frac{dy}{dt}\right)^2 = \cos^2 t + \sin^2 t + 2\sin t\cos t = 1 + \sin 2t$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = 2$$

$$A = \int_a^b x\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\; dt$$

$$= \int_0^{\frac{\pi}{2}} (\cos t + \sin t)\cdot\sqrt{2}\, dt$$

$$= \sqrt{2}\left[\sin t - \cos t\right]_0^{\frac{\pi}{2}} = \sqrt{2}\,[1 - 0 - (0 - 1)] = \sqrt{2}\,[1 + 1] = 2\sqrt{2}$$

Example 3

41 14 January 2012

Find the area of the surface obtained by rotating the circle $x^2 + (y-1)^2 = 1$ about the line $y = 0$ (5 pts)

Solution

$$x = \cos t \qquad \frac{dx}{dt} = -\sin t$$

$$y - 1 = \sin t \qquad y = 1 + \sin t \qquad \frac{dy}{dt} = \cos t$$

$$\left(\frac{dx}{dt}\right)^2 = \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = \cos^2 t$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = \cos^2 t + \sin^2 t = 1$$

$$s = 2\pi\int_a^b y\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\; dt = 2\pi\int_0^{2\pi} 1 + \sin t\, dt = 2\pi\left[t - \cos t\right]_0^{2\pi} = 2\pi[2\pi - 1 - (0 - 1)] = 4\pi^2$$

Example 4

34 August 2009 A

Consider the curve C parameterized by

$$x = \frac{1}{2}\ln(1-t^2) \quad \text{and} \quad y = \arccos t \quad \text{for} \quad 0 \le t \le 3/4$$

(a) Find the length of C

(b) Find an equation of the tangent line at the point corresponding to $t = 1/\sqrt{2}$

Solution

$$x = \frac{1}{2}\ln(1-t^2)$$

$$\frac{dx}{dt} = \frac{1}{2}\cdot\frac{-2t}{1-t^2} = \frac{-t}{1-t^2}$$

$$y = \cos^{-1} t$$

$$\frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

(a)

$$\left(\frac{dx}{dt}\right)^2 = \frac{t^2}{(1-t^2)^2}$$

$$\left(\frac{dy}{dt}\right)^2 = \frac{1}{1-t^2}$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = \frac{1}{1-t^2} + \frac{t^2}{(1-t^2)^2} = \frac{1-t^2+t^2}{(1-t^2)^2} = \frac{1}{(1-t^2)^2}$$

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$L = \int_0^{\frac{3}{4}} \sqrt{\frac{1}{(1-t^2)^2}}\, dt = \int_0^{\frac{3}{4}} \frac{1}{1-t^2}\, dt$$

$$\frac{1}{1-t^2} = \frac{1}{(1-t)(1+t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)}$$

$$\therefore A(1+t) + B(1-t) = 1$$

at $t = 1$ ⇨ $2A = 1$ $\boxed{A = \frac{1}{2}}$

at $t = -1$ ⇨ $-2B = 1$ $\boxed{B = \frac{1}{2}}$

$$L = \int_0^{\frac{3}{4}} \frac{1}{2(1-t)} + \frac{1}{2(1+t)}\, dt = \left[-\frac{1}{2}\ln(1-t) + \frac{1}{2}\ln(1+t)\right]_0^{\frac{3}{4}} = \frac{1}{2}\left[\left(-\ln\frac{1}{4} + \ln\frac{7}{4}\right) - 0\right] = \frac{1}{2}\ln 7$$

(b)

$$m = \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-1}{\sqrt{1-t^2}}\cdot\frac{1-t^2}{-t} = \frac{\sqrt{1-t^2}}{t}$$

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{1}{\sqrt{2}}} = \frac{\sqrt{1-\frac{1}{2}}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}} = 1$$

p

at $t = \frac{1}{\sqrt{2}}$

$$x\big|_{t=\frac{1}{\sqrt{2}}} = \frac{1}{2}\ln\left(1-\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln(2)$$

$$y\big|_{t=\frac{1}{\sqrt{2}}} = \cos^{-1}\frac{1}{\sqrt{2}} \quad = \frac{\pi}{4}$$

$$p\left(-\frac{1}{2}\ln(2), \frac{\pi}{4}\right) \qquad m = 1$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\pi}{4} = x + \frac{1}{2}\ln(2)$$

$$x - y + \frac{1}{2}\ln(2) + \frac{\pi}{4} = 0$$

Example 5

32 January 2009 A

Find the arc length of the curve C whose parametric equations are [3 pts.]

$$x = \ln\sqrt{1+t^2} \qquad y = \tan^{-1} t \qquad t \in [\,0\,,1\,]$$

Solution

$$x = \ln\sqrt{1+t^2} = \frac{1}{2}\ln(1+t^2) \qquad\qquad y = \tan^{-1} t$$

$$\frac{dx}{dt} = \frac{2t}{2(1+t^2)} = \frac{t}{1+t^2} \qquad\qquad \frac{dy}{dt} = \frac{1}{1+t^2}$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{t^2}{(1+t^2)^2}$$

$$\left(\frac{dy}{dt}\right)^2 = \frac{1}{(1+t^2)^2}$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = \frac{1}{(1+t^2)^2} + \frac{t^2}{(1+t^2)^2} = \frac{1+t^2}{(1+t^2)^2} = \frac{1}{(1+t^2)}$$

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$= \int_0^1 \frac{1}{\sqrt{1+t^2}}\, dt$$

let $t = \tan\theta$ $\qquad dt = \sec^2\theta \; d\theta$

at $t = 0$ ⇨ $\tan\theta = 0$ $\qquad \theta = 0$

at $t = 1$ ⇨ $\tan\theta = 1$ $\qquad \theta = \frac{\pi}{4}$

$$L = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \sec^2\theta \; d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec\theta}\; d\theta = \int_0^{\frac{\pi}{4}} \sec\theta \quad d\theta$$

$$= \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\frac{\pi}{4}} = \ln(\sqrt{2}+1) - \ln(1-0) = \ln(1+\sqrt{2})$$

Homework

1	Find the length of the curve whose parametric equations are $x = \frac{1}{2}e^{2t}$, $y = \frac{1}{3}e^{3t}$, $\ln\sqrt{2} \le t \le \ln\sqrt{8}$.	15 May 1999 A
2	Let C be a curve given parametrically by $x(t) = e^t \cos t$, $y(t) = e^t \sin t$, $0 \le t \le 2\pi$ (a) Find the points on C at which the tangent line is vertical (b) Find the length of C	16 December 1999 A
3	Let C be the curve with the parameterization : $x = 2t$ and $y = \cosh t$, $t \in [0, \ln 2]$. Find the length of C.	18 July 2000
4	Find the length of the curve C given by $x = t^2 \cos t$, $y = t^2 \sin t$; $0 \le t \le 2\pi$.	19 May 2001
5	Let C be the curve given by the parametric equations $x = 2\sin t + \sin 2t$, $y = 2\cos t + \cos 2t$, $0 \le t \le 2\pi$. (a) Find the length of C. (b) Find all point on C where the tangent line is horizontal.	31 December 2003
6	Find the length of the curve that has parametric equations $x = \frac{1}{5}t^5$ $y = \frac{1}{4}t^4$ $0 \le t \le 1$	22 June 2004 A
7	A curve C is given parametrically by $x = (t^2+1)e^t$ and $y = (t^3+2t^2)e^t$. (a) Find the points $P(x, y)$ where the tangent is vertical. (b) Determine whether C is concave upwards or downwards at $t = 0$	27 June 2006 A
8	(4+4+2 pts.) Let C be the curve given by the parametric equations $x = t - \sin t$, $y = 1 - \cos t$; $0 \le t \le \frac{\pi}{2}$ (a) Find the length of C (b) Find the area of the surface obtained by rotating C about the x −axis (c) Find $\frac{d^2y}{dx^2}$	38 Jan. 22, 2011

Homework

9	Let C be the plane curve parameterized by $x = 2t - \sin 2t \quad , \quad y = \cos 2t \quad , \quad 0 \le t \le \frac{\pi}{4}$ (a) Find the length of C. (3 pts) (b) Find the area of the surface obtained by rotating C about the $x-$axis (3 pts)	14 June 4 , 2011
10	(4+ 4 pts) Suppose a curve is given by the parametric equations $x = t - \tan^{-1} t \quad , \quad y = \sqrt{1+t^2} \quad , \quad t \in [\,0\,,1\,]$ (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (b) Find the length of the curve.	40 August 7 , 2011
11	26 January 2006 A Let Γ be a plane curve with parameterization : $x = t - e^t \quad , \quad y = t + e^t$ where $t \in \mathcal{R}$ a. Write the equation of the tangent line to Γ at the point corresponding to $t = 0$. b. Find the points on Γ (if any) where the tangent lines are vertical and those where the tangent lines are horizontal. c. Find the intervals on which this curve is concave upward and those where it is concave downward	